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OPTIMAL TRANSFER IN THE EQUATORIAL PLANE OF AN AXISYMMETRIC PLANET WITH AN ADDITIONAL CLAIM OF ACCURACY

A. N. Kovalenko

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16. Abstract

A closed system of 22 equations is derived to define the energetically optimal realization of a two-pulse transfer between elliptical orbits in the equatorial plane of an axisymmetric planet, with a correction for accuracy. Assuming we know the zero approximation found numerically or otherwise, we can accurately derive a less error-sensitive transfer, and account for the nonsphericity of the planet in question.

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OPTIMAL TRANSFER IN THE EQUATORIAL PLANE OF AN AXISYMMETRIC PLANET WITH AN ADDITIONAL CLAIM OF ACCURACY

A. N. Kovalenko

In conformity with the general theory of problem solving for optimization of pulsed transfers in studies [1]-[3] using the methods stated in the works of V. S. Novoselov [3]-[5], a closed system of 22 equations is derived to define the energetically optimal realization of a two-pulse transfer between elliptical orbits in the equatorial plane of an axisymmetric planet, with a correction for accuracy. The corrections for accuracy of performing the transfer are on the order of one less than the total energy consumption. This system can be used to specify already existing optimal simulated transfer between plane orbits. /19

1. Several Relationships Employed

The minimized functional of this study appears in the form

$$\sqrt{V_{11}^2+V_{12}^2}+\sqrt{V_{21}^2+V_{22}^2}+\sum_{i=1}^{4} P_i\sigma_i^2, \tag{1}$$

where V_{11}, V_{12} are projections of the characteristic velocity $\bar{V}_1$ onto the radius-vector and transversal for the first pulse; V_{21}, V_{22} are the corresponding values of the second pulse; σ_i^2 are the dispersions of the next values in a finite orbit: radial and transversal constituents of velocities v_r and v_ϕ, the values of the radius-vector r and the angle ϕ measured from some stationary direction; P_i -- weighted coefficients.

The values σ_i^2 are, apparently, functions of the dispersion σ^2 for values V_{11}, V_{12}, V_{21}, V_{22} (we assume it to be identical for all four characteristic velocities), and of the parameters of the transitional orbit and the force field. By analogy to study [4], we can derive a connection between σ^2 and σ^2: [sic] /20

$$
\left.
\begin{aligned}
\sigma_1^2 &= \sigma^2\left[1+\left(\frac{\partial v_r^+}{\partial v_r^-}\right)^2+\left(\frac{\partial v_r^+}{\partial v_\varphi^-}\right)^2\right],\\
\sigma_2^2 &= \sigma^2\left[1+\left(\frac{\partial v_\varphi^+}{\partial v_r^-}\right)^2+\left(\frac{\partial v_\varphi^+}{\partial v_\varphi^-}\right)^2\right],\\
\sigma_3^2 &= \sigma^2\left[\left(\frac{\partial r^+}{\partial v_r^-}\right)^2+\left(\frac{\partial r^+}{\partial v_\varphi^-}\right)^2\right],\\
\sigma_4^2 &= \sigma^2\left[\left(\frac{\partial \varphi^+}{\partial v_r^-}\right)^2+\left(\frac{\partial \varphi^+}{\partial v_\varphi^-}\right)^2\right].
\end{aligned}
\right| \qquad (2)
$$

Here and henceforth, the index 'minus' will denote the value at the start of the transitional ellipse; the index 'plus'--the end of the transitional ellipse. Isochronous derivatives on the right side of (2) for Keplerian motion exist in study [6], while for the case of the equatorial plane of an axisymmetric planet--in study [7].

The derivation of some equations of the system produced by us coincides with the statement, for instance, of study [3]. This derivation of eight limiting conditions using another two equalities for r_i, r_u (the index 'i' from now on signifies the value at the initial orbit at the instant immediately before the pulse, the index 'u'--in the ultimate orbit at the instant immediately following the pulse) are the same as four equalities derived from the condition of discontinuity of Lagrange coefficients where, it is true, the coefficients themselves must be selected for an axisymmetric planet from study [8].

These coefficients for an equatorial plane of this planet can have the form

$$
\begin{aligned}
\lambda_1 &= A\cos f+Be\sin f+CI-\alpha(t-t_0)\varkappa\frac{r_3^2e^2}{p^{7/2}}(1-e^2)^{1/2}\sin f\times\\
&\quad\times\left(3A+\frac{C}{1-e^2}\right),\\
\lambda_2 &= -A\sin f\left(1+\frac{r}{p}\right)+B\frac{p}{r}+CJ+D\frac{r}{p}+\\
&\quad+\alpha(t-t_0)\varkappa\frac{r_3^2}{p^{7/2}}e(1-e^2)^{1/2}\left\{A\left[-3\frac{p}{r}-\frac{4(1-e^2)r}{p}\right]+\right.\\
&\quad\left.+\frac{c}{1-e^2}\left[-\frac{p}{r}-\frac{3(1-e^2)r}{p}\right]\right\},\\
\lambda_3 &= -\varkappa\frac{p^{1/2}}{r^2}\left[A\sin f\frac{r}{p}-B+CK-D\frac{r}{p}\right]+
\end{aligned}
$$

$$+\alpha(t-t_0)\varkappa^2\frac{r_e^2}{rp^4}e(1-e^2)^{1/2}\left[-\left(4A+\frac{3C}{1-e^2}\right)(1-e^2)-\right.$$
$$\left.-\left(3A+\frac{C}{1-e^2}\right)\frac{p}{r}\right],$$
$$\lambda_4=-\varkappa\frac{eA}{p^{1/2}}. \qquad (3)$$

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Here A, B, C, D are arbitrary constants, subject to definition: I, J, K are Louden functions equal to [8]:

$$I=-\cos f\frac{r^3}{p^2}+2eW\sin f,$$
$$J=2\frac{p}{r}W,$$
$$K=\sin f\frac{r^2}{p^2}-2W,$$
$$W=\int_0^f\frac{\cos f\,df}{(1+e\cos f)^3}=$$
$$=\frac{\sin f}{2(1-e^2)}\left(\frac{r^2}{p^2}+\frac{r}{p}\right)-\frac{3}{2}\varkappa e\frac{t-t_\pi^0}{p^{3/2}(1-e^2)}-\frac{3}{2}\alpha(t-t_0)\varkappa e\frac{r_e^2}{p^{7/2}(1-e^2)^{1/2}}. \qquad (4)$$

Moreover in (3)-(4) the following customary notations are employed: e, p, f--eccentricity, the parameter and agitated true anomaly; κ^2--constant of gravitation; α--small dimensionless parameter, describing the nonsphericity of the planet; t--current time; t_0--some initial epoch for which the limiting osculating orbits are given; the values without an index are related to the transitional ellipse and correspond to time t; t_π--time of passage through the pericenter; r_e--equatorial radius of the planet.

Otherwise, the derivation of these 14 equations contains nothing novel and will not be discussed. Let us write only the general equations associated with it and employed henceforth--those which describe the osculating ellipse [9]:

$$v_r=\varkappa e\frac{\sin f}{p^{1/2}},\qquad \dot{p}=0,$$
$$v_\varphi=\varkappa\frac{p^{1/2}}{r},\qquad \dot{e}=0, \qquad (5)$$

$$r=\frac{p}{1+e\cos f},\qquad \dot{\omega}=\alpha\varkappa\frac{r_3^2}{p^{7/2}}(1-e^2)^{3/2},$$

$$\varphi=\omega+f,\qquad \dot{M}_0=\alpha\varkappa\frac{r_3^2}{p^{7/2}}(1-e^2)^2.$$

Here ω is the angular distance of the pericenter; M--the average anomaly.

2. Discussion of the Condition of Transversality /22

The condition of transversality is the most important part of the derivation of a system of equations. According to [2], [4] it is written as

$$\sum_{i=1}^{4}P_i\Delta\sigma_i^2+(\lambda_1\Delta v_r+\lambda_2\Delta v_\varphi+\lambda_3\Delta r+\lambda_4\Delta\varphi+H\Delta t)\Big|_{t_\text{н}}^{t_\text{к}}=0. \qquad (6)$$

The arbitrary motion along the limiting orbits is assumed. The problem consists of deriving from (6) finite relationships which can be done by switching to a set of independent variations. We will prove that these can be Δt_i, Δt_u, Δf_i, Δf_u. Let us consider a point in the initial orbit. Because we are not considering concrete motion, we are free to arbitrarily select the time of the first pulse t_i and f_i--the true anomaly of the start. From a point in the field of gravitation of the axisymmetric (and spherical) planet--and having selected t_i and f_i, we fixed that point--we can find a two-parameter family of ellipses. Let us now select t_u, and thereby we fix the ultimate ellipse and time of flyback $t_u - t_i$. And having selected f_u, we derive the second point in the plane; two the two points in the force field under investigation passes only one or more trajectories with a prescribed flyback time $t_u - t_i$.

Therefore, it has been shown that the assignment of t_i, t_u, f_i, f_u defines all other variables in (6): r_i, r_u, e, p, f^-, f^+.

Let us note that in the case of a central field, due to the immobility of the limiting orbits, we do not have to fix t_i and the number of independent variables decreases to three: f_i, f_u, $t_u - t_i = T$.

Let us reveal the condition of transversality of (6), using (5) and moving over to t_i, t_u, f_i, f_u:

$$\begin{gathered}\sum_{i=1}^{4} P_i \frac{\partial \sigma_i^2}{\partial t_\text{н}} \Delta t_\text{н} + \sum_{i=1}^{4} P_i \frac{\partial \sigma_i^2}{\partial t_\text{к}} \Delta t_\text{к} + \sum_{i=1}^{4} P_i \frac{\partial \sigma_i^2}{\partial f_\text{н}} \Delta f_\text{н} + \sum_{i=1}^{4} P_i \frac{\partial \sigma_i^2}{\partial f_\text{к}} \Delta f_\text{к} + \\ + (\lambda_1^\text{к} \varkappa e_\text{к} p_\text{к}^{-1/2} \cos f_\text{к} - \lambda_2^\text{к} \varkappa p_\text{к}^{-1/2} e_\text{к} \sin f_\text{к} + \lambda_3^\text{к} \frac{r_\text{к}^2}{p_\text{к}} e_\text{к} \sin f_\text{к} + \lambda_4^\text{к}) \Delta f_\text{к} + \\ + \lambda_4^\text{к} \alpha \varkappa \frac{r_\text{э}^2}{p_\text{к}^{7/2}} (1 - e_\text{к}^2)^{3/2} \Delta t_\text{к} - H (\Delta t_\text{к} - \Delta t_\text{н}) - \lambda_4^\text{н} \alpha \varkappa \frac{r_\text{э}^2}{p_\text{н}^{7/2}} (1 - e_\text{н}^2)^{3/2} \Delta t_\text{н} - \\ - (\lambda_1^\text{н} \varkappa e_\text{н} p_\text{н}^{-1/2} \cos f_\text{н} - \lambda_2^\text{н} \varkappa e_\text{н} p_\text{н}^{-1/2} \sin f_\text{н} + \lambda_3^\text{н} \frac{r_\text{н}^2}{p_\text{н}} e_\text{н} \sin f_\text{н} + \lambda_4^\text{н}) \Delta f_\text{н} = 0. \end{gathered} \tag{7}$$

The expressions for Δt_i, Δt_u, Δf_i, Δf_u are equal to zero with respect to the independence of variations, respectively.

Let us study the expressions

$$\frac{\partial \sigma_i^2}{\partial t_\text{н}}, \quad \frac{\partial \sigma_i^2}{\partial t_\text{к}}, \quad \frac{\partial \sigma_i^2}{\partial f_\text{н}}, \quad \frac{\partial \sigma_i^2}{\partial f_\text{к}}.$$

The value σ_i^2 according to (2) and study [7] is a function of the variables t_i, t_u, e, p, f^-, f^+.

Hence we can write the equalities /23

$$\begin{gathered} \frac{\partial \sigma_i^2}{\partial t_\text{н}} = \left(\frac{\partial \sigma_i^2}{\partial t_\text{н}}\right) + \frac{\partial \sigma_i^2}{\partial e} \frac{\partial e}{\partial t_\text{н}} + \frac{\partial \sigma_i^2}{\partial p} \frac{\partial p}{\partial t_\text{н}} + \frac{\partial \sigma_i^2}{\partial f^-} \frac{\partial f^-}{\partial t_\text{н}} + \frac{\partial \sigma_i^2}{\partial f^+} \frac{\partial f^+}{\partial t_\text{н}}, \\ \frac{\partial \sigma_i^2}{\partial t_\text{к}} = \left(\frac{\partial \sigma_i^2}{\partial t_\text{к}}\right) + \frac{\partial \sigma_i^2}{\partial e} \frac{\partial e}{\partial t_\text{к}} + \frac{\partial \sigma_i^2}{\partial p} \frac{\partial p}{\partial t_\text{к}} + \frac{\partial \sigma_i^2}{\partial f^-} \frac{\partial f^-}{\partial t_\text{к}} + \frac{\partial \sigma_i^2}{\partial f^+} \frac{\partial f^+}{\partial t_\text{к}}, \\ \frac{\partial \sigma_i^2}{\partial f_\text{н}} = \frac{\partial \sigma_i^2}{\partial e} \frac{\partial e}{\partial f_\text{н}} + \frac{\partial \sigma_i^2}{\partial p} \frac{\partial p}{\partial f_\text{н}} + \frac{\partial \sigma_i^2}{\partial f^-} \frac{\partial f^-}{\partial f_\text{н}} + \frac{\partial \sigma_i^2}{\partial f^+} \frac{\partial f^+}{\partial f_\text{н}}, \\ \frac{\partial \sigma_i^2}{\partial f_\text{к}} = \frac{\partial \sigma_i^2}{\partial e} \frac{\partial e}{\partial f_\text{к}} + \frac{\partial \sigma_i^2}{\partial p} \frac{\partial p}{\partial f_\text{к}} + \frac{\partial \sigma_i^2}{\partial f^-} \frac{\partial f^-}{\partial f_\text{к}} + \frac{\partial \sigma_i^2}{\partial f^+} \frac{\partial f^+}{\partial f_\text{к}}. \end{gathered} \tag{8}$$

The question arises on the derivation of the following matrix of derivatives:

$$\left\| \frac{\partial p,\ e,\ f^-,\ f^+}{\partial t_{\text{н}},\ t_{\text{к}},\ f_{\text{н}},\ f_{\text{к}}} \right\|.$$

Let us first calculate this matrix for the case of Keplerian motion. According to the remark on pages 4-5, let us examine three independent variables.

To calculate this matrix directly, we must know four relationships connecting the variables p, e, f^-, f^+, T, f_i and f_u. They are

$$\left.\begin{array}{c} f^+ - f^- = \omega_{\text{к}} + f_{\text{к}} - \omega_{\text{н}} - f_{\text{н}}, \quad \int\limits_{f^-}^{f^+} \frac{ds}{(1+e\cos f)^2} = \frac{\varkappa}{p^{3/2}}\, T, \\ p = \frac{p_{\text{к}}}{1+e_{\text{к}}\cos f_{\text{к}}}(1+e\cos f^+), \qquad p = \frac{p_{\text{н}}}{1+e_{\text{н}}\cos f_{\text{н}}}(1+e\cos f^-). \end{array}\right\} \tag{9}$$

Here the first relationship is the equality of the angular range of flyback to the difference of angles of the finish and start points from some direction; the second equality is the integrated integral of areas in which is posited f = f^+, t = t_u, then f = = f^-, t = t^i and the second is subtracted from the first; the third and fourth equalities are a description of the condition $r^+ = r_u$ and $r^- = r_i$.

Let us differentiate (9) with respect to T:

$$\begin{array}{c} \frac{\partial f^+}{\partial T} - \frac{\partial f^-}{\partial T} = 0, \\ r_{\text{к}}^2 \frac{\partial f^+}{\partial T} - r_{\text{н}}^2 \frac{\partial f^-}{\partial T} - 2p^2 (W^+ - W^-) \frac{\partial e}{\partial T} = -\frac{3}{2} \frac{\varkappa T}{p^{1/2}} \frac{\partial p}{\partial T} + \varkappa p^{1/2}, \\ \frac{1}{r_{\text{к}}} \frac{\partial p}{\partial T} = \cos f^+ \frac{\partial e}{\partial T} - e \sin f^+ \frac{\partial f^+}{\partial T}, \\ \frac{1}{r_{\text{н}}} \frac{\partial p}{\partial T} = \cos f^- \frac{\partial e}{\partial T} - e \sin f^- \frac{\partial f^-}{\partial T}. \end{array}$$

Here, in the second equation, the possibility is employed of removing $\partial e/\partial T$ from beneath the integral sign in view of the fact that $e = e(f_i, f_u, T)$ is not a function of f. The system produced is linear with respect to the unknowns $\frac{\partial f^+}{\partial T}, \frac{\partial f^-}{\partial T}, \frac{\partial e}{\partial T}, \frac{\partial p}{\partial T}$ /24

and is easily solved:

$$\begin{aligned} \frac{\partial f^+}{\partial T} = \frac{\partial f^-}{\partial T} &= \frac{\varkappa p^{1/2} U}{\frac{3}{2}\cdot\frac{\varkappa}{p^{1/2}} e \sin\varphi T - 2p^2 (W^+ - W^-) eS + (r_\kappa^2 - r_\text{н}^2) U}, \\ \frac{\partial e}{\partial T} &= \frac{\varkappa p^{1/2} eS}{\frac{3}{2}\frac{\varkappa}{p^{1/2}} e \sin\varphi T - 2p^2 (W^+ - W^-) eS + (r_\kappa^2 - r_\text{н}^2) U} \\ \frac{\partial p}{\partial T} &= \frac{\varkappa p^{1/2} e \sin\varphi}{\frac{3}{2}\cdot\frac{\varkappa}{p^{1/2}} e \sin\varphi T - 2p^2 (W^+ - W^-) eS + (r_\kappa^2 - r_\text{н}^2) U}. \end{aligned} \tag{10}$$

Here the notations are used:

$$\varphi = f^+ - f^-; \quad S = \frac{\sin f^+}{r_\text{н}} - \frac{\sin f^-}{r_\kappa}; \quad U = \frac{\cos f^+}{r_\text{н}} - \frac{\cos f^-}{r_\kappa}.$$

Let us differentiate (9) with respect to f_i

$$\begin{aligned} \frac{\partial f^+}{\partial f_\text{н}} - \frac{\partial f^-}{\partial f_\text{н}} &= -1, \\ r_\kappa^2 \frac{\partial f^+}{\partial f_\text{н}} - r_\text{н}^2 \frac{\partial f^-}{\partial f_\text{н}} - 2p^2 \frac{\partial e}{\partial f_\text{н}} (W^+ - W^-) + \frac{3}{2}\frac{\varkappa T}{p^{1/2}} \frac{\partial p}{\partial f_\text{н}} &= 0, \\ \frac{1}{r_\kappa}\frac{\partial p}{\partial f_\text{н}} &= \cos f^+ \frac{\partial e}{\partial f_\text{н}} - e \sin f^+ \frac{\partial f^+}{\partial f_\text{н}}, \\ \frac{1}{r_\text{н}}\frac{\partial p}{\partial f_\text{н}} &= \cos f^- \frac{\partial e}{\partial f_\text{н}} - e \sin f^- \frac{\partial f^-}{\partial f_\text{н}} + \frac{p}{p_\text{н}} e_\text{н} \sin f_\text{н}. \end{aligned}$$

Solving this system we find that

$$
\begin{aligned}
\frac{\partial f^{+}}{\partial f_{\text{н}}} &= \frac{1}{X}\left[r_{\text{н}}^{2}U - 2p^{2}(W^{+}-W^{-})\,e\,\frac{\sin f^{-}}{r_{\text{к}}} + \frac{3}{2}\varkappa T\,\frac{e\cos f^{+}\sin f^{-}}{p^{1/2}} - \right.\\
&\quad\left. - \frac{p}{p_{\text{н}}}e_{\text{н}}\sin f_{\text{н}}\left(\frac{3}{2}\varkappa\,\frac{T\cos f^{+}}{p^{1/2}} - 2p^{2}\,\frac{W^{+}-W^{-}}{r_{\text{к}}}\right)\right],\\
\frac{\partial f^{-}}{\partial f_{\text{н}}} &= \frac{1}{X}\left[r_{\text{к}}^{2}U - 2p^{2}(W^{+}-W^{-})\,e\,\frac{\sin f^{+}}{r_{\text{н}}} + \frac{3}{2}\varkappa T\,\frac{e\sin f^{+}\cos f^{-}}{p^{1/2}} - \right.\\
&\quad\left. - \frac{p}{p_{\text{н}}}e_{\text{н}}\sin f_{\text{н}}\left(\frac{3}{2}\varkappa\,\frac{T\cos f^{+}}{p^{1/2}} - 2p^{2}\,\frac{W^{+}-W^{-}}{r_{\text{к}}}\right)\right],\\
\frac{\partial e}{\partial f_{\text{н}}} &= \frac{1}{X}\left[\frac{3}{2}\varkappa\,\frac{Te^{2}}{p^{1/2}}\sin f^{-}\sin f^{+} - e\,(r_{\text{к}}\sin f^{-} - r_{\text{н}}\sin f^{+}) - \right.\\
&\quad\left. - \frac{p}{p_{\text{н}}}e_{\text{н}}\sin f_{\text{н}}\left(\frac{r_{\text{н}}^{2}-r_{\text{к}}^{2}}{r_{\text{к}}^{2}} + \frac{3}{2}\,\frac{\varkappa T}{p^{1/2}}\,e\sin f^{+}\right)\right],\\
\frac{\partial p}{\partial f_{\text{н}}} &= \frac{1}{X}\left\{2p^{2}(W^{+}-W^{-})\,e^{2}\sin f^{-}\sin f^{+} - e\,(r_{\text{к}}^{2}\sin f^{-}\cos f^{+} - \right.\\
&\quad - r_{\text{н}}^{2}\sin f^{+}\cos f^{-}) - \frac{p}{p_{\text{н}}}e_{\text{н}}\sin f_{\text{н}}\,[(r_{\text{к}}^{2}-r_{\text{н}}^{2})\cos f^{+} + \\
&\quad\left. + 2p^{2}e\sin f^{+}(W^{+}-W^{-})]\right\},
\end{aligned}
\tag{11}
$$

where X is the denominator in (10). Differentiating (9) with respect to f_u and solving the derived system, we find that /25

$$
\begin{aligned}
\frac{\partial f^{+}}{\partial f_{\text{к}}} &= \frac{1}{X}\left[-r_{\text{н}}^{2}U + 2p^{2}(W^{+}-W^{-})\,e\,\frac{\sin f^{-}}{r_{\text{к}}} - \right.\\
&\quad - \frac{3}{2}\cdot\frac{\varkappa T}{p^{1/2}}\,e\cos f^{+}\sin f^{-} + \frac{p}{p_{\text{к}}}e_{\text{к}}\sin f_{\text{к}}\times\\
&\quad\left.\times\left(\frac{3}{2}\cdot\frac{\varkappa T}{p^{1/2}}\cos f^{-} - 2p^{2}\,\frac{W^{+}-W^{-}}{r_{\text{н}}}\right)\right],\\
\frac{\partial f^{-}}{\partial f_{\text{к}}} &= \frac{1}{X}\left[-r_{\text{к}}^{2}U + 2p^{2}(W^{+}-W^{-})\,e\,\frac{\sin f^{+}}{r_{\text{н}}} - \right.\\
&\quad - \frac{3}{2}\cdot\frac{\varkappa T}{p^{1/2}}\,e\sin f^{+}\cos f^{-} + \frac{p}{p_{\text{к}}}e_{\text{к}}\sin f_{\text{к}}\times\\
&\quad\left.\times\left(\frac{3}{2}\,\frac{\varkappa T}{p^{1/2}}\cos f^{-} - 2p^{2}\,\frac{W^{+}-W^{-}}{r_{\text{н}}}\right)\right],\\
\frac{\partial e}{\partial f_{\text{к}}} &= \frac{1}{X}\left[-\frac{3}{2}\varkappa\,\frac{T}{p^{1/2}}\,e^{2}\sin f^{+}\sin f^{-} + e\,(r_{\text{к}}\sin f^{-} - r_{\text{н}}\sin f^{+}) + \right.\\
&\quad\left. + \frac{p}{p_{\text{к}}}e_{\text{к}}\sin f_{\text{к}}\left(\frac{r_{\text{н}}^{2}-r_{\text{к}}^{2}}{r_{\text{н}}^{2}} + \frac{3}{2}\cdot\frac{\varkappa T}{p^{1/2}}\,e\sin f^{-}\right)\right],\\
\frac{\partial p}{\partial f_{\text{к}}} &= \frac{1}{X}\left[-2p^{2}(W^{+}-W^{-})\,e^{2}\sin f^{-}\sin f^{+} + \right.\\
&\quad + e\,(r_{\text{к}}^{2}\sin f^{-}\cos f^{+} - r_{\text{н}}^{2}\sin f^{+}\cos f^{-}) + \\
&\quad\left. + \frac{p}{p_{\text{к}}}e_{\text{к}}\sin f_{\text{к}}\,[(r_{\text{н}}^{2}-r_{\text{к}}^{2})\cos f^{-} + 2p^{2}e\sin f^{-}(W^{+}-W^{-})]\right\};
\end{aligned}
\tag{12}
$$

Formulas (10), (11), (12) produce the matrix $\left\|\frac{\partial p,\, e,\, f^{-},\, f^{+}}{\partial T,\, f_{\text{н}},\, f_{\text{к}}}\right\|$ for Keplerian motion. Because the derivatives appearing in this matrix, according to (8), (2), (7) will be multiplied by $P_i\sigma^2$, it is sufficient to know it for Keplerian motion and for the solution of the problem with accuracy to within the first negative power. Moreover, we can indicate a means of calculating the matrix $\left\| \frac{\partial p,\, e,\, f^-,\, f^+}{\partial t_{\text{н}},\, t_{\text{к}},\, f_{\text{н}},\, f_{\text{к}}} \right\|$ even for flyback in the equatorial plane of an axisymmetric planet. For this purpose, we must differentiate instead of (9) the analogous equations in a new force field:

$$\left.\begin{aligned} & f^+ + \alpha(t_{\text{к}} - t_{\text{н}})\varkappa\frac{r_{\text{э}}^2}{p^{7/2}}(1-e^2)^{3/2} - f^- = \\ & = \omega_{\text{к}0} + \alpha(t_{\text{к}} - t_0)\frac{\varkappa r_{\text{э}}^2}{p_{\text{к}}^{7/2}}(1-e_{\text{к}}^2)^{3/2} - \omega_{\text{н}0} + f_{\text{к}} - \\ & - \alpha(t_{\text{н}} - t_0)\frac{\varkappa r_{\text{э}}^2}{p_{\text{н}}^{7/2}}(1-e_{\text{н}}^2)^{3/2} - f_{\text{н}}, \\ & \int_{f^-}^{f^+} \frac{df}{(1+e\cos f)^2} = \frac{\varkappa}{p^{3/2}}\left[t_{\text{к}} - t_{\text{н}} + \frac{\alpha r_{\text{э}}^2}{p^2}(1-e^2)^{1/2}(t_{\text{к}} - t_{\text{н}})\right], \\ & p(1+e_{\text{к}}\cos f_{\text{к}}) = p_{\text{к}}(1+e\cos f^+), \\ & p(1+e_{\text{н}}\cos f_{\text{н}}) = p_{\text{н}}(1+e\cos f^-). \end{aligned}\right\}$$

The linear systems obtained after differentiation are solved with /26
respect to the terms of the unknown matrix.

The knowledge of the matrix $\left\| \frac{\partial p,\, e,\, f^-,\, f^+}{\partial t_{\text{н}},\, t_{\text{к}},\, f_{\text{н}},\, f_{\text{к}}} \right\|$ (or the matrix $\left\| \frac{\partial p,\, e,\, f^-,\, f^+}{\partial T,\, f_{\text{н}},\, f_{\text{к}}} \right\|$), makes it possible to derive from the condition of transversality in the form of (6) finite relationships for the arbitrary coplar elliptical orbits in the central field, and also for elliptical orbits in the equatorial plane of an axisymmetric planet.

3. Writing the System of Equations

Let us return to (7) and write the finite relationships mentioned on its basis and on the basis of (8).

Equating the bracket to zero at Δt_i produces

$$\begin{aligned} -H = {} & \sum_{i=1}^{4} P_i \frac{\partial \sigma_i^2}{\partial t_{\text{н}}} + \frac{\partial e}{\partial t_{\text{н}}} \sum_{i=1}^{4} P_i \frac{\partial \sigma_i^2}{\partial e} + \frac{\partial p}{\partial t_{\text{н}}} \sum_{i=1}^{4} P_i \frac{\partial \sigma_i^2}{\partial p} + \\ & + \frac{\partial f^-}{\partial t_{\text{н}}} \sum_{i=1}^{4} P_i \frac{\partial \sigma_i^2}{\partial f^-} + \frac{\partial f^+}{\partial t_{\text{н}}} \sum_{i=1}^{4} P_i \frac{\partial \sigma_i^2}{\partial f^+} - \alpha\lambda_4^{\text{н}}\varkappa \frac{r_{\text{э}}^2}{p_{\text{н}}^{7/2}}(1-e_{\text{н}}^2)^{3/2}; \end{aligned} \qquad (13)$$

the bracket at t_u--

$$H=\sum_{i=1}^{4}P_i\frac{\partial\sigma_i^2}{\partial t_\text{к}}+\frac{\partial e}{\partial t_\text{к}}\sum_{i=1}^{4}P_i\frac{\partial\sigma_i^2}{\partial e}+\frac{\partial p}{\partial t_\text{к}}\sum_{i=1}^{4}P_i\frac{\partial\sigma_i^2}{\partial p}+$$
$$+\frac{\partial f^-}{\partial t_\text{к}}\sum_{i=1}^{4}P_i\frac{\partial\sigma_i^2}{\partial f^-}+\frac{\partial f^+}{\partial t_\text{к}}\sum_{i=1}^{4}P_i\frac{\partial\sigma_i^2}{\partial f^+}+\alpha\lambda_4^\text{к}\varkappa\frac{r_\text{э}^2}{p_\text{н}^{7/2}}(1-e_\text{н}^2)^{3/2}; \tag{14}$$

the bracket at f_i--

$$\frac{\partial e}{\partial f_\text{н}}\sum_{i=1}^{4}P_i\frac{\partial\sigma_i^2}{\partial e}+\frac{\partial p}{\partial f_\text{н}}\sum_{i=1}^{4}P_i\frac{\partial\sigma_i^2}{\partial p}+\frac{\partial f^-}{\partial f_\text{н}}\sum_{i=1}^{4}P_i\frac{\partial\sigma_i^2}{\partial f^-}+$$
$$+\frac{\partial f^+}{\partial f_\text{н}}\sum_{i=1}^{4}P_i\frac{\partial\sigma_i^2}{\partial f^+}-(\lambda_1^\text{н}\varkappa e_\text{н}p_\text{н}^{-1/2}\cos f_\text{н}-\lambda_2^\text{н}\varkappa e_\text{н}p_\text{н}^{-1/2}\sin f_\text{н}+$$
$$+\lambda_3^\text{н}\frac{r_\text{н}^2}{p_\text{н}}e_\text{н}\sin f_\text{н}+\lambda_4^\text{н})=0; \tag{15}$$

the bracket at f_u--

$$\frac{\partial e}{\partial f_\text{к}}\sum_{i=1}^{4}P_i\frac{\partial\sigma_i^2}{\partial e}+\frac{\partial p}{\partial f_\text{к}}\sum_{i=1}^{4}P_i\frac{\partial\sigma_i^2}{\partial p}+\frac{\partial f^-}{\partial f_\text{к}}\sum_{i=1}^{4}P_i\frac{\partial\sigma_i^2}{\partial f^-}+$$
$$+\frac{\partial f^+}{\partial f_\text{к}}\sum_{i=1}^{4}P_i\frac{\partial\sigma_i^2}{\partial f^+}+\lambda_1^\text{к}\varkappa e_\text{к}p_\text{к}^{-1/2}\cos f_\text{к}-\lambda_2^\text{к}\varkappa e_\text{к}p_\text{к}^{-1/2}\sin f_\text{к}+$$
$$+\lambda_3^\text{к}\frac{r_\text{к}^2}{p_\text{к}}e_\text{к}\sin f_\text{к}+\lambda_4^\text{к}=0. \tag{16}$$

In equations (13)-(16), the derivatives of σ_i^2 are calculated in terms of (2) and study [7]; the derivatives of e, p, f^-, and f^+ are calculated for the case of Keplerian motion according to (10)-(12); the Lagrange coefficients are taken from (3) with the corresponding indexes 'i' and 'u'. /27

Herein we must add the value of H, calculated according to its definition (cf. [8]):

$$H=-\varkappa^2 ep^{-2}C. \tag{17}$$

Let us add to (13)-(17) the remaining conditions of optimal fly-back.

The limiting conditions:

$$\varkappa e p^{-1/2} \sin f^- - \varkappa e_{\text{н}} p_{\text{н}}^{-1/2} \sin f_{\text{н}} = V_1 \cos \psi_1, \tag{18}$$

$$\varkappa e p^{-1/2} \sin f^+ - \varkappa e_{\text{к}} p_{\text{к}}^{-1/2} \sin f_{\text{к}} = -V_2 \cos \psi_2, \tag{19}$$

$$\varkappa p^{1/2} r_{\text{н}}^{-1} - \varkappa p_{\text{н}}^{1/2} r_{\text{н}}^{-1} = V_1 \sin \psi_1, \tag{20}$$

$$\varkappa p^{1/2} r_{\text{к}}^{-1} - \varkappa p_{\text{к}}^{1/2} r_{\text{к}}^{-1} = -V_2 \sin \psi_2, \tag{21}$$

$$p_{\text{н}} (1 + e \cos f^-) = p (1 + e_{\text{н}} \cos f_{\text{н}}), \tag{22}$$

$$p_{\text{к}} (1 + e \cos f^+) = p (1 + e_{\text{к}} \cos f_{\text{к}}), \tag{23}$$

$$f^- + \omega_0 + \alpha (t_{\text{н}} - t_0) \frac{\varkappa r_{\text{з}}^2}{p^{7/2}} (1 - e^2)^{3/2} = f_{\text{н}} + \omega_{\text{н}0} + \alpha (t_{\text{н}} - t_0) \frac{\varkappa r_{\text{з}}^2 (1 - e_{\text{н}}^2)}{p_{\text{н}}^{7/2}}, \tag{24}$$

$$f^+ + \omega_0 + \alpha (t_{\text{к}} - t_0) \frac{\varkappa r_{\text{з}}^2}{p^{7/2}} (1 - e^2)^{3/2} = f_{\text{к}} + \omega_{\text{к}0} + \alpha (t_{\text{к}} - t_0) \frac{\varkappa r_{\text{з}}^2 (1 - e_{\text{к}}^2)}{p_{\text{к}}^{7/2}}. \tag{25}$$

$$r_{\text{н}} = p (1 + e \cos f^-)^{-1}, \tag{26}$$

$$r_{\text{к}} = p (1 + e \cos f^+)^{-1}. \tag{27}$$

Here ψ_1, ψ_2 are the angles between $\overline{V}_1$, $\overline{V}_2$ and the radius-vector.

The conditions of discontinuity of the Lagrange coefficients:

$$A \cos f^- + Be \sin f^- + CI^- - \alpha (t_{\text{н}} - t_0) \frac{\varkappa r_{\text{з}}^2}{p^{7/2}} e^2 (1 - e^2)^{1/2} \times \left(3A + \frac{C}{1 - e^2}\right) = \cos \psi_1, \tag{28}$$

$$A\cos f^{+}+Be\sin f^{+}+CI^{+}-\alpha(t_{\text{к}}-t_{0})\frac{\varkappa r_3^2}{p^{7/2}}e^2(1-e^2)^{1/2}\times$$
$$\times\left(3A+\frac{C}{1-e^2}\right)=\cos\psi_2, \quad (29)$$

$$-A\left(1+\frac{r_{\text{н}}}{p}\right)\sin f^{-}+B\frac{p}{r_{\text{н}}}+CJ^{-}+D\frac{r_{\text{н}}}{p}+$$
$$+\alpha(t_{\text{н}}-t_{0})\frac{\varkappa r_3^2}{p^{7/2}}\frac{e}{(1-e^2)^{1/2}}\left\{A\left[-3\frac{p}{r_{\text{н}}}-4(1-e^2)\frac{r_{\text{н}}}{p}\right]+\right.$$
$$\left.+\frac{C}{1-e^2}\left[-\frac{p}{r_{\text{н}}}-3(1-e^2)\frac{r_{\text{н}}}{p}\right]\right\}=\sin\psi_1, \quad (30)$$

$$-A\left(1+\frac{r_{\text{к}}}{p}\right)\sin f^{+}+B\frac{p}{r_{\text{к}}}+CJ^{+}+D\frac{r_{\text{к}}}{p}+$$
$$+\alpha(t_{\text{к}}-t_{0})\frac{\varkappa r_3^2}{p^{7/2}}\frac{e}{(1-e^2)^{1/2}}\left\{A\left[-3\frac{p}{r_{\text{к}}}-4(1-e^2)\frac{r_{\text{к}}}{p}\right]+\right.$$
$$\left.+\frac{C}{1-e^2}\left[-\frac{p}{r_{\text{к}}}-3(1-e^2)\frac{r_{\text{к}}}{p}\right]\right\}=\sin\psi_2. \quad (31)$$

The connection of time with the true anomaly is given through the /28 introduction of eccentric anomaly E and the Kepler equation:

$$\operatorname{tg}\frac{E^{-}}{2}=\left(\frac{1-e}{1+e}\right)^{1/2}\operatorname{tg}\frac{f^{-}}{2}, \quad (32)$$

$$\operatorname{tg}\frac{E^{+}}{2}=\left(\frac{1-e}{1+e}\right)^{1/2}\operatorname{tg}\frac{f^{+}}{2}, \quad (33)$$

$$E^{+}-e\sin E^{+}=$$
$$=E^{-}-e\sin E^{-}+\alpha(t_{\text{к}}-t_{\text{н}})\frac{\varkappa r_3^2}{p^{7/2}}(1-e^2)^2+\frac{\varkappa}{p^{3/2}}(1-e^2)^{3/2}(t_{\text{к}}-t_{\text{н}}). \quad (34)$$

The system (13)-(34) is a system of 22 equations with 22 unknowns:

$$p,\ e,\ \omega_0,\ f^{-},\ f^{+},\ f_{\text{н}},\ f_{\text{к}},\ r_{\text{н}},\ r_{\text{к}},\ V_1,\ V_2,$$
$$\psi_1,\ \psi_2,\ A,\ B,\ C,\ D,\ H,\ t_{\text{н}},\ t_{\text{к}},\ E^{-},\ E^{+}.$$

Some values in (13)-(34) are small, which is quite useful in deriving the system. The correction for accuracy is small versus the total energy losses and this circumstance is useful in deriving formulas (2). Also small is the parameter of nonsphericity α, which is used in several places in section 1 for derivation of formulas (3), (5) and others.

But if in (3) and (5) we assume the values λ_i, e, p, ω, M calculated with accuracy to within the first negative power exclusively, then we must consider the value $\alpha(t_u - t_0)$ and not α to be small, for otherwise its contribution to the first power in these values bears a more complete account of nonsphericity than is done in this study (axisymmetric planet). The last comment excludes the use, within the framework of this study, of the alluring selection of such moment t_i (or t_u) at which the limiting orbits will come into coaxial juxtaposition, affording energy savings.

Being transcendental, system (13)-(34) is extremely difficult for general solution. We now do not even know its solution under the assumption $\alpha = 0$, $P_i = 0$. The question arises as to the expediency of deriving such a system. Here we can indicate the importance of using it to specify existing optimal simulated transfer. That is, assuming we know the zero approximation found numerically or otherwise, and substituting this in (13)-(34), we produce the problem of a precision derivation near the given simulated transfer of a second, less error-sensitive transfer for implementing the pulses. In this context, it is also possible to account for nonsphericity. In addition to the concrete derivation of this solution, the problem of a general study of the neighborhood of the given simulated transfer for accuracy of implementation is possible, also using system (13)-(34). The concrete exe- /29
cution of such precision is most simple to demonstrate in the neighborhood of the Homann ellipse.

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